Effects of the Systematic Nonlinear Space charge stopbands on High Intensity Accelerators

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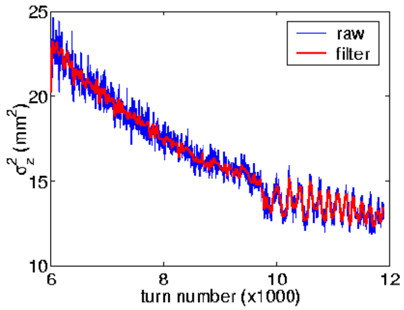
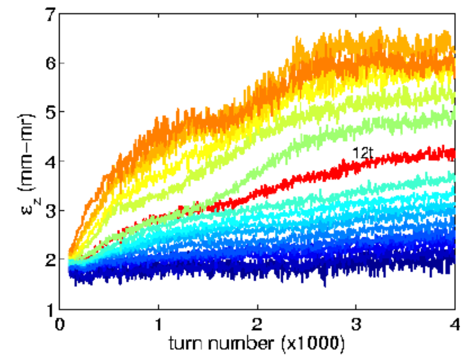
1. Quick review of Fermilab Booster Modeling [PRSTAB **9**, 014202 (2006)]
2. SNS-type accelerators
3. FFAG accelerators
4. Conclusion

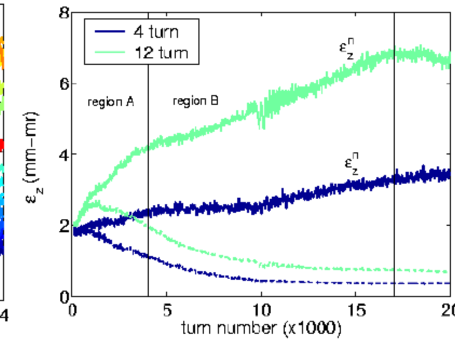
All unhappy accelerators have their emittance blowup mechanisms



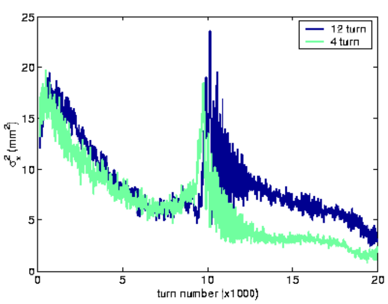
Space charge effects: Linac delivers about 30 mA beam current to the Fermilab Booster, i.e. 4.2×10^{11} particles in one injection turn. Machine modeling: $\beta x=6.3$, $\beta z=21.4$, $Dx=2.54$, at the IPM location with $Qx=6.7$, $Qz=6.8$.

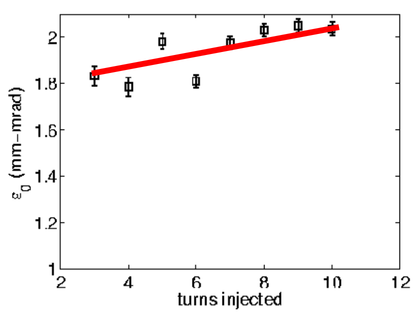
$$\varepsilon = a_0 + b_1 t + b_2 \int_0^t K_{sc}\, dt \qquad K_{sc} = \frac{2Nr_0}{\beta^2\gamma^3}$$





Horizontal IPM measurements





$$\sigma_x^2 = \beta_x\epsilon_{\rm rms} + D^2\sigma_\delta^2 = \beta_x\frac{\epsilon^n_{\rm rms}}{\beta\gamma}\frac{\beta_{x0}(\beta\gamma)_0}{\beta_{x0}(\beta\gamma)_0} + D^2\sigma_{\delta0}^2\frac{\sigma_\delta^2}{\sigma_{\delta0}^2} = aA(t) + bB(t)$$

$$a = \epsilon^n_{\rm rms}\frac{\beta_{x0}}{\beta_0\gamma_0}, \quad A(t) = \frac{\beta_x}{\beta_{x0}}\frac{\beta_0\gamma_0}{\beta\gamma}$$

$$b = D^2\sigma_{\delta0}^2, \quad B(t) = \frac{\gamma_0\sqrt{\gamma_0|\eta_0|/V_0|\cos\phi_{s0}|}}{\gamma\sqrt{\gamma|\eta|/V|\cos\phi_s|}}, \qquad \sigma_x^2 = (a_0 + a_1 t)A(t) + b_0 B(t).$$

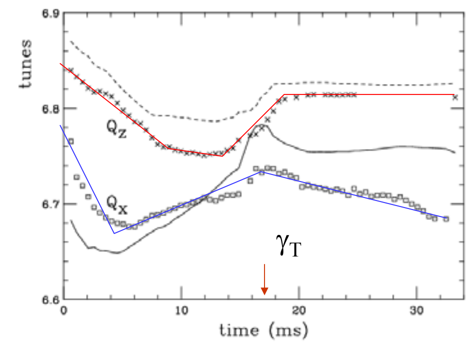
Summary:
1. $d\varepsilon_z/dt \sim K_{sc}$
2. ε_z increases linearly with t at about 1 π-mm-mrad in 10^4 revolutions.
3. The horizontal emittance and the off-momentum spread can be separated by using different scaling law (energy dependence).
4. The horizontal emittance is less affected by the space charge force! Why
5. The slow linear growth of the horizontal emittance is the same as that of the vertical plane!
6. The post-transition horizontal bunch-width oscillation is induced essentially by the longitudinal mis-match of the bunch shape with rf potential well. Using the bunch shaoe mis-match, one can deduce the phase space area.

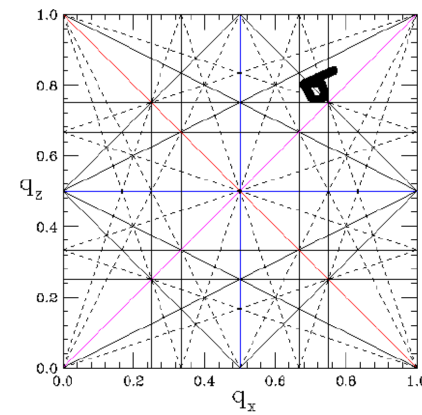
Modeling algorithm:

We consider N-particle in Gaussian distribution and construct a model with 24 Superperiod FODO cells

$$M_{\mathrm{D}\to\mathrm{F}} = \begin{pmatrix} \sqrt{\frac{\beta_{x,\mathrm{F}}}{\beta_{x,\mathrm{D}}}}\cos\psi_x & \sqrt{\beta_{x,\mathrm{F}}\beta_{x,\mathrm{D}}}\sin\psi_x & 0 & 0 \\ -\frac{1}{\sqrt{\beta_{x,\mathrm{F}}\beta_{x,\mathrm{D}}}}\sin\psi_x & \sqrt{\frac{\beta_{x,\mathrm{D}}}{\beta_{x,\mathrm{F}}}}\cos\psi_x & 0 & 0 \\ 0 & 0 & \sqrt{\frac{\beta_{z,\mathrm{F}}}{\beta_{z,\mathrm{D}}}}\cos\psi_z & \sqrt{\beta_{z,\mathrm{F}}\beta_{z,\mathrm{D}}}\sin\psi_z \\ 0 & 0 & -\frac{1}{\sqrt{\beta_{z,\mathrm{F}}\beta_{z,\mathrm{D}}}}\sin\psi_z & \sqrt{\frac{\beta_{z,\mathrm{D}}}{\beta_{z,\mathrm{F}}}}\cos\psi_z \end{pmatrix}$$

$$M_{\mathrm{F}\to\mathrm{D}} = \begin{pmatrix} \sqrt{\frac{\beta_{x,\mathrm{D}}}{\beta_{x,\mathrm{F}}}}\cos\psi_x & \sqrt{\beta_{x,\mathrm{F}}\beta_{x,\mathrm{D}}}\sin\psi_x & 0 & 0 \\ -\frac{1}{\sqrt{\beta_{x,\mathrm{F}}\beta_{x,\mathrm{D}}}}\sin\psi_x & \sqrt{\frac{\beta_{x,\mathrm{F}}}{\beta_{x,\mathrm{D}}}}\cos\psi_x & 0 & 0 \\ 0 & 0 & \sqrt{\frac{\beta_{z,\mathrm{D}}}{\beta_{z,\mathrm{F}}}}\cos\psi_z & \sqrt{\beta_{z,\mathrm{F}}\beta_{z,\mathrm{D}}}\sin\psi_z \\ 0 & 0 & -\frac{1}{\sqrt{\beta_{z,\mathrm{F}}\beta_{z,\mathrm{D}}}}\sin\psi_z & \sqrt{\frac{\beta_{z,\mathrm{F}}}{\beta_{z,\mathrm{D}}}}\cos\psi_z \end{pmatrix}$$





Space charge force is a local kick on every half cell:

$$\rho(x,z) = \frac{Ne}{2\pi\sigma_x\sigma_z}\exp\{-\frac{x^2}{2\sigma_x^2} - \frac{z^2}{2\sigma_z^2}\},$$

$$\begin{aligned} V(x,z) &= \frac{Nr_0}{\beta^2\gamma^3}\int_0^\infty \frac{1-\exp\{-\frac{x^2}{2\sigma_x^2+t} - \frac{z^2}{2\sigma_z^2+t}\}}{\sqrt{(2\sigma_x^2+t)(2\sigma_z^2+t)}}dt \\ &\approx \frac{Nr_0}{\beta^2\gamma^3}\left(\frac{x^2}{\sigma_x(\sigma_x+\sigma_z)} + \frac{z^2}{\sigma_z(\sigma_x+\sigma_z)}\right) \\ &\quad -\frac{Nr_0}{4\beta^2\gamma^3\sigma_x^2(\sigma_x+\sigma_z)^2}\left(\frac{2+R}{3}x^4 + \frac{2}{R}x^2z^2 + \frac{1+2R}{3R^3}z^4\right) + \cdots \end{aligned}$$

$$\begin{aligned} \Delta x' &= -\frac{\partial V}{\partial x}\ell \approx \frac{2Nr_0\ell}{\beta^2\gamma^3\sigma_x(\sigma_x+\sigma_z)}x\exp\{-\frac{x^2+z^2}{(\sigma+\sigma_z)^2}\}, \\ \Delta z' &= -\frac{\partial V}{\partial z}\ell \approx \frac{2Nr_0\ell}{\beta^2\gamma^3\sigma_z(\sigma_x+\sigma_z)}z\exp\{-\frac{x^2+z^2}{(\sigma+\sigma_z)^2}\}, \end{aligned}$$

- Sextupole nonlinearity on each half cell for nonlinearity in dipoles
- Linear coupling,
- Random quadrupoles with zero tune shifts
- Random closed orbit error
- Dynamical aperture of 80 by 50 pi-mm-mrad

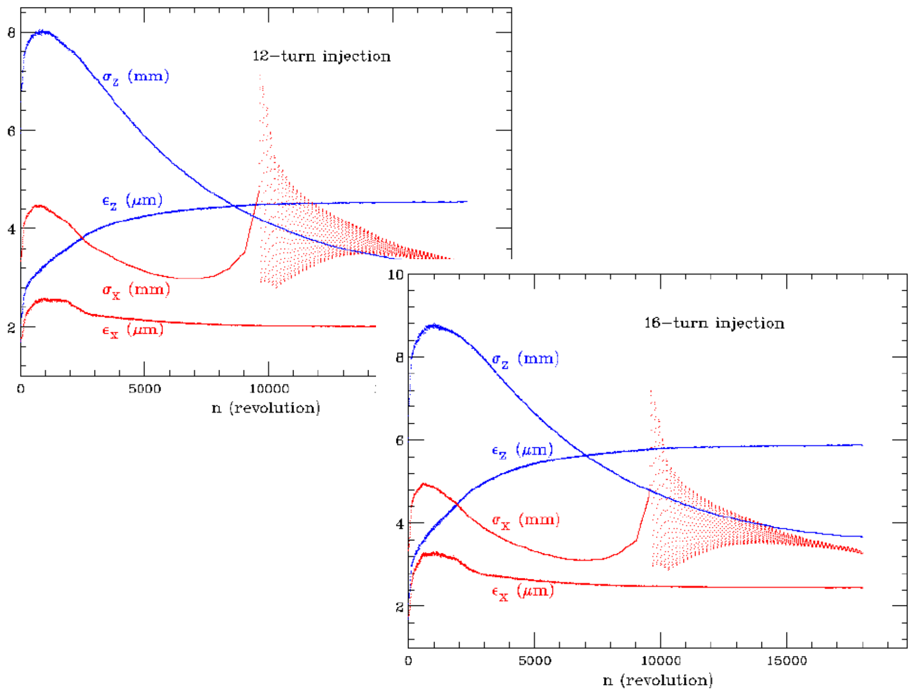
$$x'' + K_x(s)x = \frac{b_0(s)}{\rho} + \frac{b_1(s)}{\rho}x + \frac{a_1(s)}{\rho}z + \frac{1}{2}\frac{b_2(s)}{\rho}(x^2 - z^2)$$

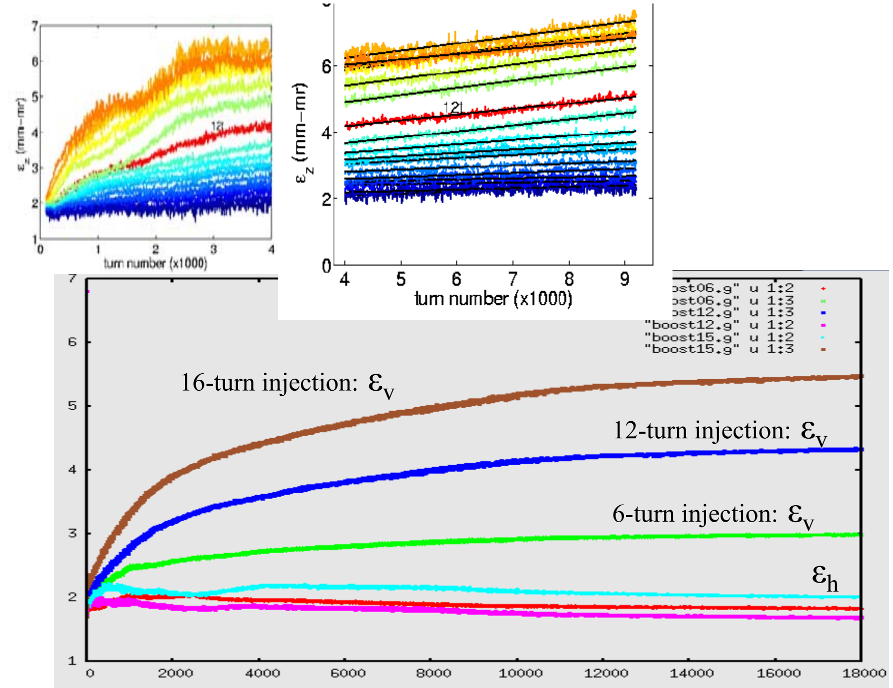
$$z'' + K_z(s)z = -\frac{a_0(s)}{\rho} - \frac{b_1(s)}{\rho}z + \frac{a_1(s)}{\rho}x - \frac{b_2(s)}{\rho}xz$$

Random number generators are used to generate b0, a0, b1, and a1. The quadrupole error is subject to a constraint with zero tune shift. The integrated sextupole strengths are set to the systematic values: -0.0173 m^{-2} and -0.263 m^{-2} for focusing and defocusing dipoles respectively.

$$\begin{aligned} \delta_{\mathrm{rms}}(n) &= \delta_{\mathrm{rms}}(1)B_f(n)\left(1 + (G_\delta - 1)(1 - \exp(-\alpha_g(n-n_t)))\right) \\ &\quad [1 + A_\delta\exp(-\alpha_g(n-n_t))\sin(2\pi(n-n_t)f)], \end{aligned}$$

(For $n>n_t$=9600) G_δ=2, A_δ=0.5, f=1/150, α_δ=1/(15*150)





2. What is the effect of the nonlinear systematic space charge resonances on beam emittances?

The space charge potential has the form of $\exp(-(x^2+z^2)/4\sigma^2)$. We know that the Montque resonance is produced by the x^2z^2 term in the potential. How about the systematic resonance induced by the terms x^4, z^4, x^2z^2, x^6, x^4z^2, x^2z^4, z^6, etc? Since the space charge potential follows the beam profile, which has the same superperiodicity, systematic resonances are located at 4qx=P, 4qz=P, 2qx+2qz=P, 6qx=P, 6qz=P, etc.

What is the effects of systematic resonances?

1) S. Machida, NIMA 384, 316 (1997)
2) Ingo Hofmann, Giuliano Franchetti, and Alexei V. Fedotov , HB2002, AIP conference proceedings
3) S. Igarashi et al., observed at the KEKPS at injection, PAC2003, p.2610 (2003)
4) Oliver Boine-Frankenheim observed the 4th order resonance in bunch rotation at the SIS18 simulations.

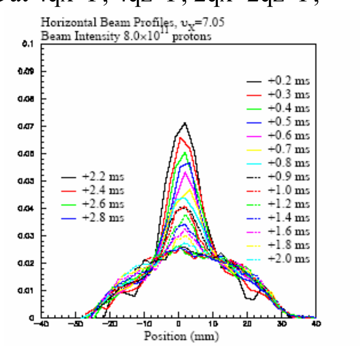
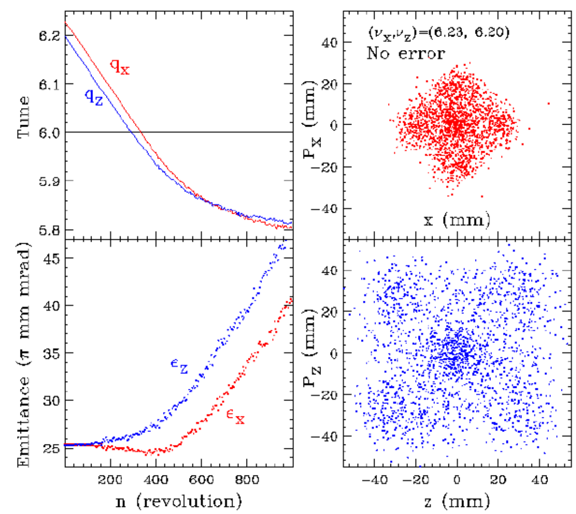


Figure 1: Horizontal beam profiles 0.2 ~ 2.8 ms after injection when the horizontal tune was 7.05 and the injection beam intensity was 8.0×10^{11} protons.

If one chooses the bare tunes at (6.23, 6.20), and 1000 injection-turns, with a total tune-shift of 1.1. **Note that the tune shift of SNS is only 0.15!**

Phase space map at the end of injection.

The tune for small amplitude particles continue to decrease as the particle is injected!



$$V(x,z) = \frac{K_{sc}}{2}\int_0^\infty \frac{-1+\exp\{-\frac{x^2}{2\sigma_x^2+t}-\frac{z^2}{2\sigma_z^2+t}\}}{\sqrt{(2\sigma_x^2+t)(2\sigma_z^2+t)}}dt$$

$$\approx -\frac{K_{sc}}{2}\left\{\left(\frac{x^2}{\sigma_x(\sigma_x+\sigma_z)}+\frac{z^2}{\sigma_z(\sigma_x+\sigma_z)}\right) - \frac{1}{8\sigma_x^2(\sigma_x+\sigma_z)^2}\left(\frac{2+R}{3}x^4+\frac{2}{R}x^2z^2+\frac{1+2R}{3R^3}z^4\right)+\cdots\right\}$$

$$V_{sc,4}(x,z) = -\frac{K_{sc}}{16\sigma_x^2(\sigma_x+\sigma_z)^2}\left(\frac{2+R}{3}x^4+\frac{2}{R}x^2z^2+\frac{1+2R}{3R^3}z^4\right)$$

$$\begin{aligned} V_{sc,4}(J_x,J_z,\psi_x,\psi_z,\theta) \approx &-\sum|G_{4,0,\ell}|J_x^2\cos(4\psi_x-\ell\theta+\chi_{4,0,\ell}) \\ &-\sum|G_{0,4,\ell}|J_z^2\cos(4\psi_z-\ell\theta+\chi_{0,4,\ell}) \\ &-\sum|G_{2,2,\ell}|J_xJ_z\cos(2\psi_x+2\psi_z-\ell\theta+\chi_{2,2,\ell}) \\ &-\sum|G_{2,-2,\ell}|J_xJ_z\cos(2\psi_x-2\psi_z-\ell\theta+\chi_{2,-2,\ell}), \end{aligned}$$

$$G_{4,0,\ell} = \frac{1}{2\pi}\oint\frac{K_{sc}\beta_x^2(2\sigma_x+\sigma_z)}{48\sigma_x^3(\sigma_x+\sigma_z)^2}\exp\{j(4\phi_x-4\nu_x\theta+\ell\theta)\}ds,$$

$$G_{0,4,\ell} = \frac{1}{2\pi}\oint\frac{K_{sc}\beta_z^2(\sigma_x+2\sigma_z)}{48\sigma_z^3(\sigma_x+\sigma_z)^2}\exp\{j(4\phi_z-4\nu_z\theta+\ell\theta)\}ds,$$

$$G_{2,2,\ell} = \frac{1}{2\pi}\oint\frac{K_{sc}\beta_x\beta_z}{16\sigma_x\sigma_z(\sigma_x+\sigma_z)^2}\exp\{j(2\phi_x+2\phi_z-2\nu_x\theta-2\nu_z\theta+\ell\theta)\}ds,$$

$$G_{2,-2,\ell} = \frac{1}{2\pi}\oint\frac{K_{sc}\beta_x\beta_z}{16\sigma_x\sigma_z(\sigma_x+\sigma_z)^2}\exp\{j(2\phi_x-2\phi_z-2\nu_x\theta+2\nu_z\theta+\ell\theta)\}ds,$$

FODO cells

$$\phi_x = \int_0^s\frac{ds}{\beta_x},\quad \phi_z = \int_0^s\frac{ds}{\beta_z}.$$

$G_{4,0,\ell} = 1.48\times10^{-2}(K_{sc}C/8\pi\epsilon^2)$

$G_{0,4,\ell} = -1.49\times10^{-2}(K_{sc}C/8\pi\epsilon^2)$

$G_{2,2,\ell} = -1.04\times10^{-4}(K_{sc}C/8\pi\epsilon^2)$

$G_{2,-2,\ell} = 1.17\times10^{-1}(K_{sc}C/8\pi\epsilon^2)$

2-kick approximation: 0.00702, −0.00604, −0.000576, and 0.0982.

See also: S Machida, Nucl. Inst. Methods A384, 316 (1997)

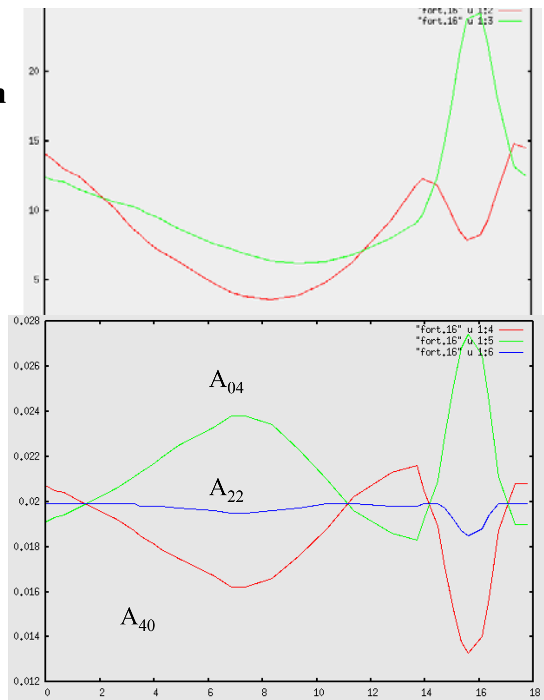
SIS18
space charge octupole term

$G_{4,0,\ell}$=1.25E-3

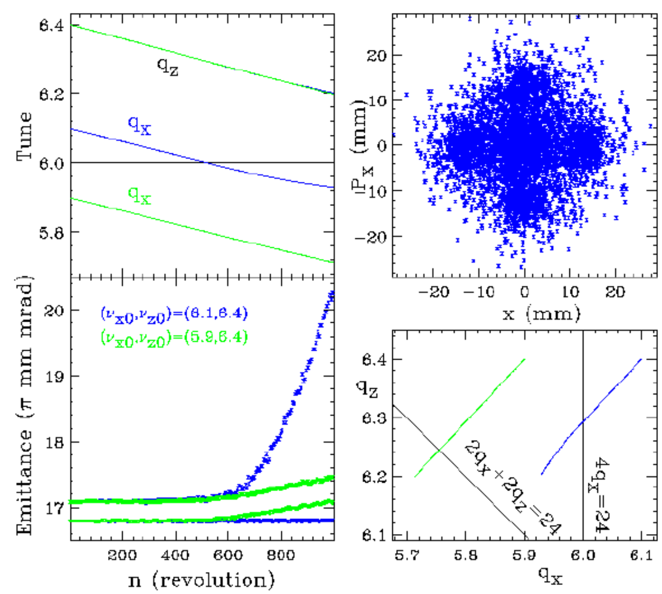
$G_{0,4,\ell}$=1.33E-3

$G_{2,2,\ell}$=2.62E-4 ←

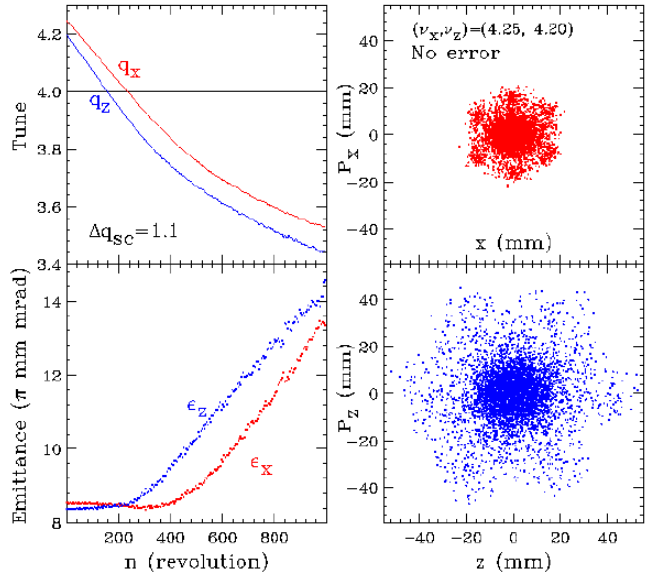
$G_{2,-2,0}$=0.124



Effect of 4qx=24, and 2qx+2qz=24:



How about the 6qx=P and 6qz=P resonances?



3. Non-scaling FFAG

design by A. Ruggiero, AP-Technote: 219 (BNL-ADD)

Thanks to Shinji's talk at GSI!

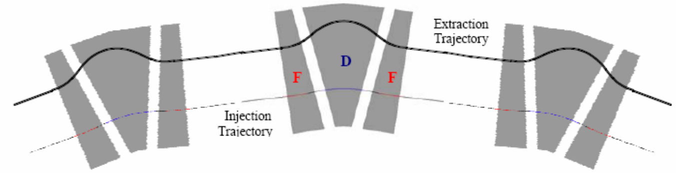


Table 1. Major Parameters of the 3 FFAG Rings (Proton Driver)

		Inj. Ring	*LE Ring*	*HE Ring*
Energy: Inj.	GeV	0.40	1.50	4.45
Ext.		1.50	4.45	11.6
β Inj.		0.7131	0.9230	0.9847
Ext.		0.9230	0.9847	0.9972
Δp/p	±%	40.45	40.43	40.41
Circumference	m	807.091	818.960	830.829
No. of Periods		136	136	136
Period Length	m	5.934	6.022	6.109
Harmonic No.		136	138	140
RF Inj.	MHz	36.02	46.03	49.75
Ext.	λ = 5.9345 m	46.03	49.75	50.38

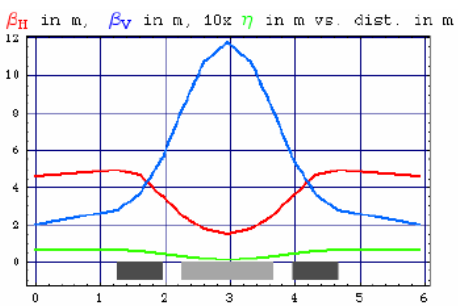


Figure 4. Lattice Functions along the Length of a Period

Table 3. Global Lattice Parameters

Phase Adv. / Cell	H	105.234°
	V	99.9395°
Betatron Tune,	H	39.755
	V	37.755
Nat. Chromaticity,	H	-0.9263
	V	-1.8052
Transition Energy,	γ_T	105.482 i

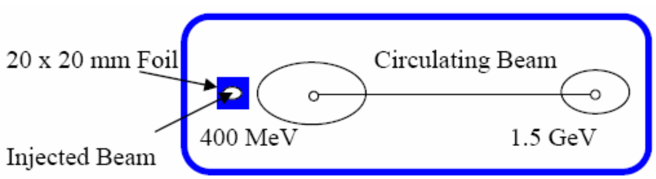


Figure 5. Beam Footprint in the Injection area with a 30 x 10 cm Vacuum Chamber

Table 5. Space-Charge, Beam Size and Beam Intensity

		Inj. Ring	*LE Ring*	*HE Ring*
Protons / pulse		1.0×10^{14}	1.0×10^{14}	1.0×10^{14}
Average Beam Current	mA	1.60	1.60	1.60
Average Beam Power	MW	2.40	7.12	18.56
Full Nor. Emittance	π mm-mrad	100	100	100
Actual Inj. Emittance	π mm-mrad	98.32	41.69	17.68
Bunching Factor		4.0	4.0	4.0
Tune-Shift		0.343	0.188	0.085
Half Vert. Beam Size	cm	2.12	1.38	0.90
Half Hor. Beam Size	cm	3.41	2.22	1.46

Table 6. RF Acceleration Parameters for the Pulsed Mode

		Inj. Ring	*LE Ring*	*HE Ring*
Energy Gain per Turn	MeV/turn	0.60	0.90	2.00
No. of Revolutions		1834	3278	3576
RF Peak Voltage	MVolt	1.20	1.80	4.00
Acceleration Period	ms	6.137	9.398	10.001
Injection Period	ms	1.144	--	--
Max. Repetition Rate	kHz	0.137	0.106	0.100
Gap Voltage	kVolt	20	30	40
Gaps / Cavity		2	2	2
Number of Cavities		30	30	50

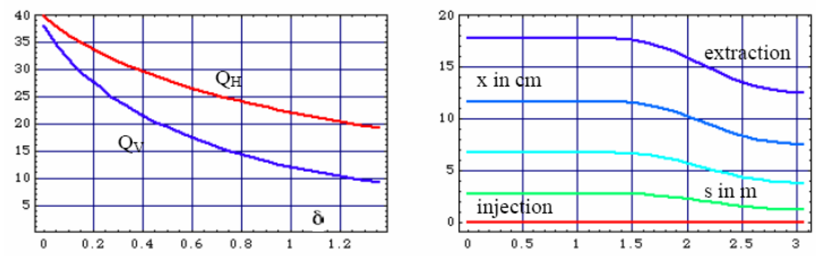
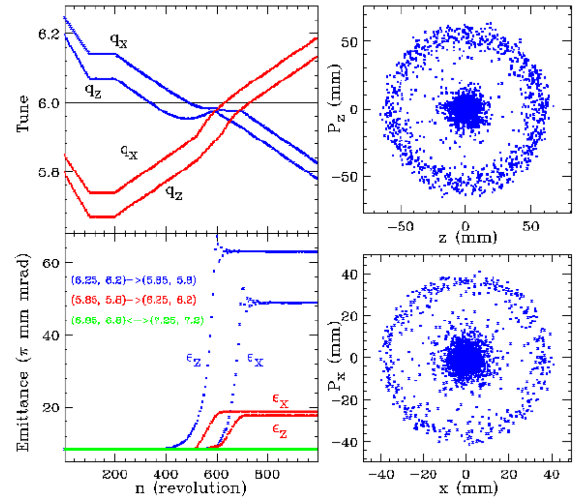
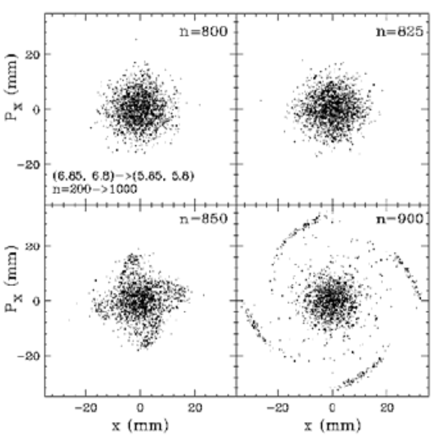
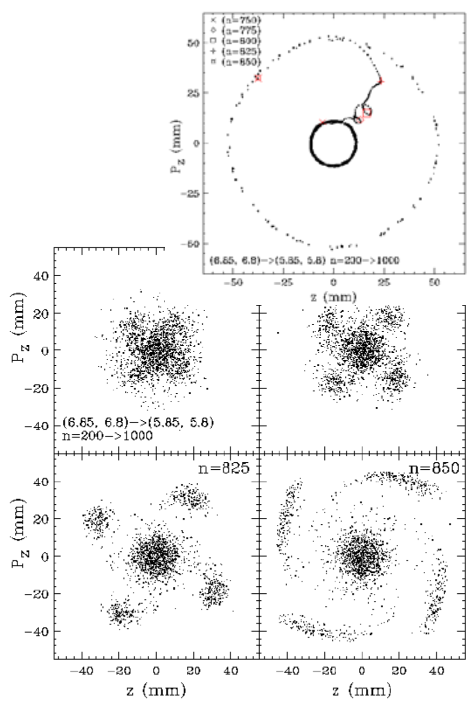


Figure 10. Tune Variation vs. Momentum   Figure 11. Closed Orbits vs Half-Period Length

$dQx/dn=20/2000=0.01$
$dQz/dn=30/2000=0.015$



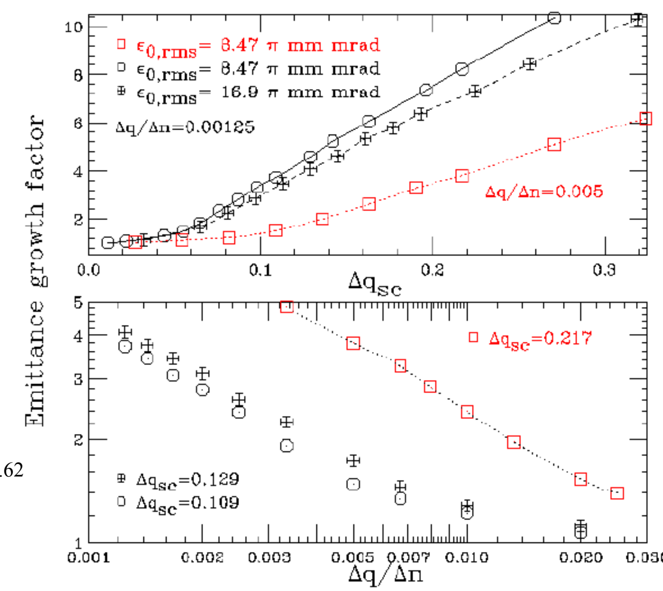




EGF=ε(final)/ε(initial)

EGF~($\Delta\nu_{sc}$)

EGF~$(dq/dn)^{-0.62}$



Gradient errors: $y'' + K(s)y + k(s)y = 0$

Floquet transformation: $\eta = \frac{y}{\sqrt{\beta_y}},\quad \varphi = \frac{1}{\nu}\int_0^s \frac{ds}{\beta_y}$ $\qquad \ddot\eta + \nu^2\eta + \nu^2\beta_y^2 k(s)\eta = 0$

$$\nu\beta_y^2 k(s) = \sum J_p e^{jp\varphi},\quad J_p = \frac{1}{2\pi}\oint \nu\beta_y^2 k(s) e^{-jp\varphi} d\varphi = \frac{1}{2\pi}\oint \beta_y k(s) e^{-jp\varphi} ds$$

$$\ddot\eta + \nu^2\eta + \nu\sum J_p e^{jp\varphi}\eta \approx \ddot\eta + \nu^2\eta + 2\nu g_p\cos(p\varphi+\chi_p)\eta = 0$$

$$H = \tfrac12\dot\eta^2 + \tfrac12\nu^2\eta^2 + \nu g_p \cos(p\varphi+\chi_p)\eta^2$$

Mathieu's equation

$$\eta = \frac{\sqrt{2J}}{\sqrt{\nu}}\cos\psi,$$

$$H = \nu J + J g_p\cos(p\varphi+\chi_p)(1+\cos 2\psi) \approx \nu J + \tfrac12 J g_p \cos(2\psi - p\varphi - \chi_p)$$

$$F_2 = (\psi - \tfrac12 p\varphi - \tfrac12\chi_p)I \qquad H = (\nu - \tfrac12 p)I + \tfrac12 I g_p\cos(2\phi)$$

$$\ddot I = g_p^2 I + 2[g_p(\nu - \tfrac12 p)\cos 2\phi]I$$

At ν=p/2: $\quad I = ae^{g_p\varphi} + be^{-g_p\varphi} \sim e^{2\pi g_p n}$

Sum resonances driven by the skew quads:

$$x'' + K_x(s)x + 2gz' - (q-g')z = 0,$$
$$z'' + K_z(s)z + 2gx' - (q+g')x = 0,$$

$$G_{1,\mp1,\ell}\, e^{j\chi_\mp} = \frac{1}{2\pi}\oint \sqrt{\beta_x\beta_z}\, A_{lc\mp}(s)\, e^{j[\chi_x \mp \chi_z - (\nu_x \mp \nu_z - \ell)\theta]} ds.$$

$$A_{lc\mp}(s) = -\frac{a_1}{\rho} + g(s)\left(\frac{\alpha_x}{\beta_x} - \frac{\alpha_z}{\beta_z}\right) + jg(s)\left(\frac{1}{\beta_x} \pm \frac{1}{\beta_z}\right),$$

$$H = \nu_x J_x + \nu_z J_z + g\sqrt{J_x J_z}\cos(\psi_x + \psi_z - \ell\theta + \chi), \qquad g = |G_{1,1,\ell}|$$

$$F_2 = (\psi_x + \psi_z - \ell\theta + \chi)I_1 + \psi_z I_2 \qquad I_1 = J_x, \quad I_2 = J_z - J_x$$

$$H = \delta I_1 + g\sqrt{I_1(I_1+I_2)}\cos\phi_1 + \nu_z I_2, \qquad \delta = (\nu_x + \nu_z - \ell)$$

$$\ddot I_1 = [g^2 - \delta^2]I_1 + (\tfrac12 g^2 I_2 + \delta H)$$

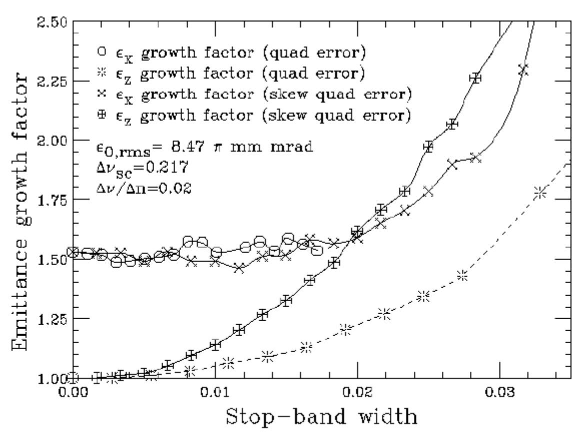
$$I_1 = ae^{\sqrt{g^2-\delta^2}\theta} + be^{-\sqrt{g^2-\delta^2}\theta} + c + d\theta + \tfrac12(\tfrac12 g^2 I_2 + \delta H)\theta^2$$

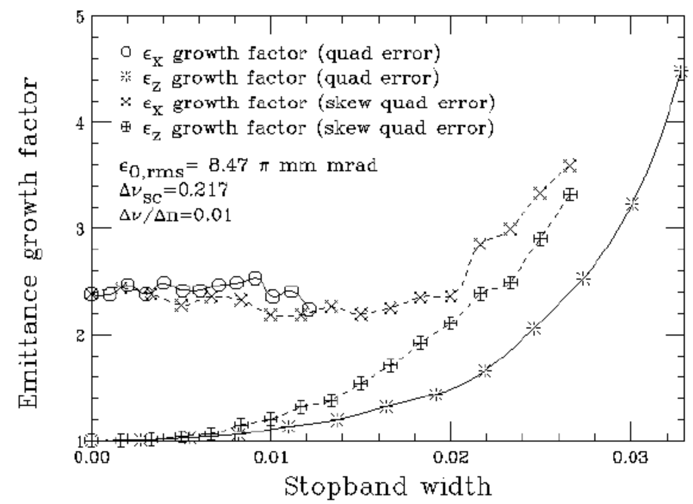
$$\sim ae^{2\pi\sqrt{g^2-\delta^2}n}$$

When the tunes ramp through resonances, the number of turns that the tune stays on resonance is Δn=g/(dν/dn). Thus we expect that the emittance growth is given by

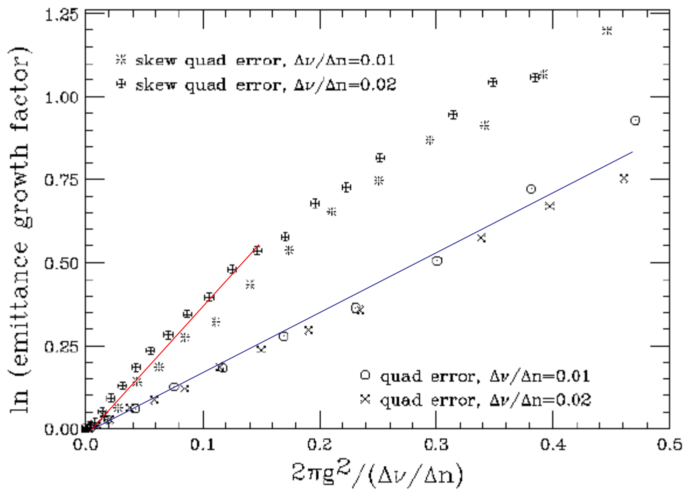
$$\mathrm{EGF} = \exp\left[\lambda \frac{2\pi g^2}{d\nu/dn}\right]$$

In this calculation, the tunes are ramped from (6.85,7.80) to (5.85, 6.80)





λ≈1.5 for quadrupole error, and 3.5 for skew quadrupole error



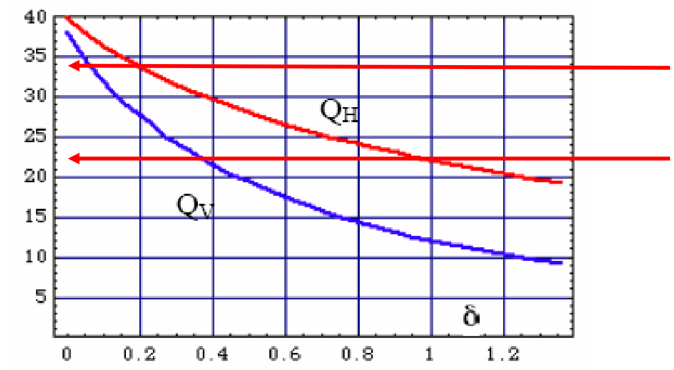


Figure 10. Tune Variation vs. Momentum

136/4=34
136/6=22.6666

Conclusion:

1. Systematic Nonlinear space charge resonances can be important in high intensity accelerators
2. For future neutron source design, one should try to avoid the systematic nonlinear space charge resonance, if it is possible!
3. For the non-scaling FFAG, the nonlinear resonances induced by the space charge potential can be the limiting factor. These resonances limit the phase advance of each basic cell to within π/2 to π/3, and thus the momentum acceptance is highly constrained.
4. I also find that the emittance growth factor for quadrupole and skew-quadrupole errors obeys a simple scaling law:

$$\mathrm{EGF} = \exp\left[\frac{\lambda 2\pi g^2}{d\nu / dn}\right]$$