Effects of the Systematic Nonlinear Space charge stopbands on High Intensity Accelerators

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1. Quick review of Fermilab Booster Modeling
   [PRSTAB 9, 014202 (2006)]

2. SNS-type accelerators
3. FFAG accelerators
4. Conclusion

"All unhappy accelerators have their emittance blowup mechanisms"

Space charge effects: Linac delivers about 30 mA beam current to the Fermilab Booster, i.e. \(4.2 \times 10^{11}\) particles in one injection turn. Machine modeling: \(\beta_x=6.3, \beta_z=21.4, D_x=2.54\), at the IPM location with \(Q_x=6.7, Q_z=6.8\).

\[
\varepsilon = a_0 + b_1 t + b_2 \int_0^t K_{sc} \, dt
\]

\[
K_{sc} = \frac{2N r_0}{\beta^3} \gamma^3
\]

Horizontal IPM measurements

Summary:
1. \(d\varepsilon_z/dt \sim K_{sc}\)
2. \(\varepsilon_z\) increases linearly with \(t\) at about \(1 \pi\)-mm-mrad in \(10^4\) revolutions.
3. The horizontal emittance and the off-momentum spread can be separated by using different scaling law (energy dependence).
4. The horizontal emittance is less affected by the space charge force! Why
5. The slow linear growth of the horizontal emittance is the same as that of the vertical plane!
6. The post-transition horizontal bunch-width oscillation is induced essentially by the longitudinal mis-match of the bunch shape with rf potential well. Using the bunch shape mis-match, one can deduce the phase space area.

\[
\sigma_z^2 = \beta_x e_{rms}^2 + D^2 \sigma_0^2 = \frac{e_{rms} e_{0 \gamma}}{\beta \gamma} + D^2 \sigma_0^2 = a A(t) + b B(t)
\]

\[
a = \frac{e_{rms}}{e_0}, \quad b = D^2 \sigma_0^2, \quad A(t) = \frac{e_{0 \gamma}}{\beta \gamma} \frac{1}{\gamma^2}, \quad B(t) = \frac{\gamma_0 (e_0) \cos \phi_0}{\gamma \cos \phi_0}, \quad \sigma_z^2 = (a_0 + a_1 t) A(t) + b_0 B(t).
\]
Modeling algorithm:
We consider N-particle in Gaussian distribution and construct a model with 24 Superperiod FODO cells

\[
M_{D\rightarrow F} = \begin{pmatrix}
-\frac{\beta_p^2 \rho_x^2 \cos \psi_x}{\beta_p^2 \rho_x^2 \sin \psi_x} & \frac{\beta_p^2 \rho_x \rho_y \sin \psi_x}{\beta_p^2 \rho_x \rho_y \cos \psi_x} & 0 & 0 \\
-\frac{1}{\beta_p^2 \rho_x} \sin \psi_x & \frac{\beta_p^2 \rho_x \cos \psi_x}{\beta_p^2 \rho_x \sin \psi_x} & 0 & 0 \\
0 & 0 & \frac{\beta_p^2 \rho_y \cos \psi_x}{\beta_p^2 \rho_y \sin \psi_x} & \frac{\beta_p^2 \rho_y \sin \psi_x}{\beta_p^2 \rho_y \cos \psi_x} \\
0 & 0 & -\frac{1}{\beta_p^2 \rho_y} \sin \psi_x & \frac{\beta_p^2 \rho_y \cos \psi_x}{\beta_p^2 \rho_y \sin \psi_x}
\end{pmatrix}
\]

\[
M_{P\rightarrow D} = \begin{pmatrix}
-\frac{\beta_p^2 \rho_x^2 \cos \psi_x}{\beta_p^2 \rho_x^2 \sin \psi_x} & \frac{\beta_p^2 \rho_x \rho_y \sin \psi_x}{\beta_p^2 \rho_x \rho_y \cos \psi_x} & 0 & 0 \\
-\frac{1}{\beta_p^2 \rho_x} \sin \psi_x & \frac{\beta_p^2 \rho_x \cos \psi_x}{\beta_p^2 \rho_x \sin \psi_x} & 0 & 0 \\
0 & 0 & \frac{\beta_p^2 \rho_y \cos \psi_x}{\beta_p^2 \rho_y \sin \psi_x} & \frac{\beta_p^2 \rho_y \sin \psi_x}{\beta_p^2 \rho_y \cos \psi_x} \\
0 & 0 & -\frac{1}{\beta_p^2 \rho_y} \sin \psi_x & \frac{\beta_p^2 \rho_y \cos \psi_x}{\beta_p^2 \rho_y \sin \psi_x}
\end{pmatrix}
\]

Space charge force is a local kick on every half cell:
\[
\rho(x, z) = \frac{N e}{2\pi \sigma_x \sigma_z} \exp\left\{ \frac{x^2}{2\sigma_x^2} - \frac{z^2}{2\sigma_z^2} \right\}
\]

\[
V(x, z) = \frac{N r_0}{\sigma_x^2 \sigma_z} \int_0^\infty \frac{1 - \exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{z^2}{2\sigma_z^2}\right)}{\sqrt{2\pi \sqrt{t}}} dt \\
\approx \frac{N r_0}{\sigma_x^2 \sigma_z} \frac{1}{\left\{\sigma_x (\sigma_x + \sigma_z) + \sigma_z (\sigma_x + \sigma_z)\right\}^2} \left[ 2R^2 x^2 + \frac{2}{3} \sigma_x^2 z^2 + \frac{1 + 2R^2 z^2}{3R^2} \right] + \ldots
\]

\[
\Delta x' = \frac{\partial V}{\partial x} \approx \frac{2N r_0 \ell}{\beta \gamma^3 \sigma_z (\sigma_z + \sigma_x)} x \exp\left\{ -\frac{x^2 + z^2}{(\sigma_z + \sigma_x)^2} \right\},
\]

\[
\Delta z' = \frac{\partial V}{\partial z} \approx \frac{2N r_0 \ell}{\beta \gamma^3 \sigma_z (\sigma_z + \sigma_x)} z \exp\left\{ -\frac{x^2 + z^2}{(\sigma_z + \sigma_x)^2} \right\},
\]

- Sextupole nonlinearity on each half cell for nonlinearity in dipoles
- Linear coupling,
- Random quadrupoles with zero tune shifts
- Random closed orbit error
- Dynamical aperture of 80 by 50 pi-mm-mrad

\[
x'' + K_+ (s)x = \frac{b_0(s)}{\rho} + \frac{b_1(s)}{\rho} x + \frac{a_1(s)}{\rho} z + \frac{1}{2} \frac{b_2(s)}{\rho} (x^2 - z^2)
\]

\[
z'' + K_+ (s)z = -\frac{a_0(s)}{\rho} - \frac{b_1(s)}{\rho} x + \frac{a_1(s)}{\rho} z - \frac{b_2(s)}{\rho} x z
\]

Random number generators are used to generate b0, a0, b1, and a1. The quadrupole error is subject to a constraint with zero tune shift. The integrated sextupole strengths are set to the systematic values: -0.0173 m^-2 and -0.263 m^-2 for focusing and defocusing dipoles respectively.

\[
\delta_{rms}(n) = \delta_{rms}(1) B_f(n) \left[ 1 + (G_d - 1)(1 - \exp(-\alpha_d (n - n_i))) \right] [1 + A_d \exp(-\alpha_d (n - n_i)) \sin(2\pi (n - n_i) / f)]
\]

(For n_{n_i}=9600) \ G_d=2, \ A_d=0.5, \ f=1/150, \ \alpha_d=1/(15*150)
2. What is the effect of the nonlinear systematic space charge resonances on beam emittances?

The space charge potential has the form of \( \exp(-x^2z^2/4\sigma^2) \). We know that the Montque resonance is produced by the \( x^2z^2 \) term in the potential. How about the systematic resonance induced by the terms \( x^4, z^4, x^2z^2, x^6, z^6, \) etc? Since the space charge potential follows the beam profile, which has the same superperiodicity, systematic resonances are located at \( 4qx=P, 4qz=P, 2qx+2qz=P, 6qx=P, 6qz=P, \) etc.

What is the effects of systematic resonances?
1) S. Machida, NIMA 384, 316 (1997)
2) Ingo Hofmann, Giuliano Franchetti, and Alexei V. Fedotov, HB2002, AIP conference proceedings
3) S. Igarashi et al., observed at the KEKPS at injection, PAC2003, p.2610 (2003)
4) Oliver Boine-Frankenheim observed the 4th order resonance in bunch rotation at the SIS18 simulations.

If one chooses the bare tunes at (6.23, 6.20), and 1000 injection-turns, with a total tune-shift of 1.1.

**Note that the tune shift of SNS is only 0.15!**

The tune for small amplitude particles continue to decrease as the particle is injected.

Phase space map at the end of injection.
$$V(x, z) = \frac{K_{sc}}{2} \int_0^\infty \frac{-1 + \exp\left\{\frac{-x^2}{2\sigma_x^2 + t} - \frac{x^2}{2\sigma_z^2 + t}\right\} dt}{\sqrt{(2\sigma_x^2 + t)(2\sigma_z^2 + t)}}$$

$$\approx - \frac{K_{sc}}{2} \left\{ \left( \frac{x^2}{\sigma_x^2(\sigma_x + \sigma_z)} + \frac{x^2}{\sigma_z(\sigma_x + \sigma_z)} \right) \right\}$$

$$\chi_{4,0,\ell} = \frac{1}{2\pi} \int K_{sc} \frac{\beta_x^4}{\sigma_x^2(\sigma_x + \sigma_z)^2} \exp\left\{j(4\phi_x - 4\nu_x\theta + 4\ell\varphi)\right\} ds,$$

$$\chi_{0,4,\ell} = \frac{1}{2\pi} \int K_{sc} \frac{\beta_x^2(2\sigma_x + 2\sigma_z)}{4\sigma_x^2(\sigma_x + \sigma_z)^2} \exp\left\{j(4\phi_x - 4\nu_x\theta + 4\ell\varphi)\right\} ds,$$

$$\chi_{2,2,\ell} = \frac{1}{2\pi} \int K_{sc} \frac{\beta_x^2(2\sigma_x + 2\sigma_z)}{16\sigma_x(\sigma_x + \sigma_z)^2} \exp\left\{j(2\phi_x - 2\phi_z - 2\nu_x\theta + 2\nu_z\theta + 2\ell\varphi)\right\} ds,$$

$$\chi_{2,-2,\ell} = \frac{1}{2\pi} \int K_{sc} \frac{\beta_x^2(2\sigma_x + 2\sigma_z)}{16\sigma_x(\sigma_x + \sigma_z)^2} \exp\left\{j(2\phi_x - 2\phi_z - 2\nu_x\theta + 2\nu_z\theta + 2\ell\varphi)\right\} ds,$$

2-kick approximation: 0.00702, -0.00604, -0.00576, and 0.0982.

How about the $6q_x=P$ and $6q_z=P$ resonances?

3. Non-scaling FFAG
design by A. Ruggiero, AP-Technote: 219 (BNL-ADD)

Table 1. Major Parameters of the 3 FFAG Rings (Proton Driver)

<table>
<thead>
<tr>
<th></th>
<th>Inj. Ring</th>
<th>LER Ring</th>
<th>HRE Ring</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy:</td>
<td></td>
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<tr>
<td>Injected</td>
<td>0.40</td>
<td>1.50</td>
<td>4.45</td>
</tr>
<tr>
<td>Extracted</td>
<td>1.50</td>
<td>4.45</td>
<td>11.6</td>
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<tr>
<td>$\beta$</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Injected</td>
<td>0.7131</td>
<td>0.9230</td>
<td>0.9847</td>
</tr>
<tr>
<td>Extracted</td>
<td>0.9230</td>
<td>0.9847</td>
<td>0.9972</td>
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<tr>
<td>$\Delta p/p$</td>
<td>$\pm4%$</td>
<td>$40.45%$</td>
<td>$40.43%$</td>
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<tr>
<td>Circumference</td>
<td>m</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>807.091</td>
<td>818.960</td>
<td>830.829</td>
</tr>
<tr>
<td>No. of Periods</td>
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<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>136</td>
<td>136</td>
<td>136</td>
</tr>
<tr>
<td>Period Length</td>
<td>m</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>5.9344</td>
<td>6.022</td>
<td>6.109</td>
</tr>
<tr>
<td>Harmonic No.</td>
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<td></td>
<td>136</td>
<td>138</td>
<td>140</td>
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<tr>
<td>RF</td>
<td>MHz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Injected</td>
<td>36.02</td>
<td>46.03</td>
<td>49.75</td>
</tr>
<tr>
<td>Extracted</td>
<td>$\omega = 5.9345$ m</td>
<td>46.03</td>
<td>49.75</td>
</tr>
</tbody>
</table>

Table 3. Global Lattice Parameters

| Phase Adv. / Cell | H | 105.234° | 105.234° |
| Betatron Tune,    | H | 39.755   | 37.755   |
| Nat. Chromaticity | H | -0.9263  | -1.8052  |
| Transition Energy  | $\omega$ | 105.482  |

Table 5. Space-Charge, Beam Size and Beam Intensity

<table>
<thead>
<tr>
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<th>Inj. Ring</th>
<th>LER Ring</th>
<th>HRE Ring</th>
</tr>
</thead>
<tbody>
<tr>
<td>Protons / pulse</td>
<td>$1.0 \times 10^5$</td>
<td>$1.0 \times 10^4$</td>
<td>$1.0 \times 10^3$</td>
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<tr>
<td>Average Beam Current</td>
<td>mA</td>
<td>1.60</td>
<td>1.60</td>
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<tr>
<td>Average Beam Power</td>
<td>MW</td>
<td>2.40</td>
<td>7.12</td>
</tr>
<tr>
<td>Full Nor. Emittance</td>
<td>$\pi$ mm-mrad</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Actual Inj. Emittance</td>
<td>$\pi$ mm-mrad</td>
<td>98.32</td>
<td>41.69</td>
</tr>
<tr>
<td>Bunching Factor</td>
<td>$\omega$</td>
<td>4.0</td>
<td>4.0</td>
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<tr>
<td>Tune-Shift</td>
<td>$\omega$</td>
<td>0.345</td>
<td>0.188</td>
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<tr>
<td>Half Vert. Beam Size</td>
<td>$\omega$</td>
<td>2.12</td>
<td>1.28</td>
</tr>
<tr>
<td>Half Hor. Beam Size</td>
<td>$\omega$</td>
<td>3.41</td>
<td>2.22</td>
</tr>
</tbody>
</table>

Table 6. RF Acceleration Parameters for the Pulsed Mode

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<th>Inj. Ring</th>
<th>LER Ring</th>
<th>HRE Ring</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy Gain per Turn</td>
<td>MeV/turn</td>
<td>0.60</td>
<td>0.90</td>
</tr>
<tr>
<td>No. of Revolutions</td>
<td>18.34</td>
<td>3278</td>
<td>3576</td>
</tr>
<tr>
<td>RF Peak Voltage</td>
<td>MVolt</td>
<td>1.20</td>
<td>1.80</td>
</tr>
<tr>
<td>Acceleration Period</td>
<td>ms</td>
<td>6.137</td>
<td>9.398</td>
</tr>
<tr>
<td>Injection Period</td>
<td>ms</td>
<td>1.144</td>
<td>--</td>
</tr>
<tr>
<td>Max. Repetition Rate</td>
<td>kHz</td>
<td>0.137</td>
<td>0.106</td>
</tr>
<tr>
<td>Gap Voltage</td>
<td>kVolt</td>
<td>2.0</td>
<td>3.0</td>
</tr>
<tr>
<td>Gaps / Cavity</td>
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<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Number of Cavities</td>
<td></td>
<td>30</td>
<td>30</td>
</tr>
</tbody>
</table>

Figure 4. Lattice Functions along the Length of a Period

Figure 5. Beam Footprint in the Injection area with a 30 x 10 cm Vacuum Chamber
\[
\frac{dQ_x}{dn} = \frac{20}{2000} = 0.01
\]
\[
\frac{dQ_z}{dn} = \frac{30}{2000} = 0.015
\]

\[
\text{EGF} \sim \left(\frac{\Delta \nu_{sc}}{n} \right)^{-0.62}
\]

\[
\text{EGF} = \frac{\varepsilon(\text{final})}{\varepsilon(\text{initial})}
\]
Gradient errors: \[ y'' + K(s)y + k(s)y = 0 \]

Floquet transformation: 
\[ \eta = \frac{y}{\sqrt{\beta}}, \quad \phi = \frac{1}{\nu} \frac{ds}{\beta}, \quad \dot{\eta} + \nu^2 \eta + \nu^2 \beta^2 k(s) \eta = 0 \]

\[ \nu \beta^2 k(s) = \sum J_p e^{i\nu p}, \quad J_p = \frac{1}{2\pi} \frac{1}{\nu} \nu^2 \beta^2 k(s) e^{-i\nu p} d\phi = \frac{1}{2\pi} \int \beta(s) e^{-i\nu p} ds \]

\[ \dot{\eta} + \nu^2 \eta + \nu \sum J_p e^{i\nu p} \eta \approx \frac{\dot{\eta} + \nu^2 \eta + 2\nu g_p \cos(p\phi + \chi_p) \eta = 0}{} \]

\[ H = \frac{1}{2} \dot{\eta}^2 + \frac{1}{2} \nu^2 \eta^2 + \nu g_p \cos(p\phi + \chi_p) \eta^2 \]

Mathieu's equation
\[ \eta = \frac{\sqrt{2J}}{\sqrt{\nu}} \cos \psi, \]

\[ H = \nu J + Jg_p \cos(p\phi + \chi_p) \eta + \cos 2\psi \approx \nu J + \frac{1}{2} Jg_p \cos(2\psi - p\phi - \chi_p) \]

\[ F_2 = (\psi - \frac{1}{2} p\phi - \frac{1}{2} \chi_p) J \quad H = (\nu - \frac{1}{2} p) I + \frac{1}{2} Jg_p \cos(2\phi) \]

\[ \ddot{I} = g_p^2 I + 2 [g_p (\nu - \frac{1}{2} p) \cos 2\phi] I \]

At \(\nu = p/2\): 
\[ I = ae^{g_p \phi} + be^{-g_p \phi} \sim e^{2\nu g_p \phi} \]

Sum resonances driven by the skew quads:
\[ x'' + K_x(s)x + 2g'z' - (q - g')z = 0, \]

\[ z'' + K_s(s)z + 2g'x' - (q + g')x = 0, \]

\[ G_{1,1,1,1} e^{i2\nu x} = \frac{1}{2\pi} \int \sqrt{\beta_x \beta_z} A_{1,1}(s) \; e^{i2\nu x} \; e^{i\chi_x} \; e^{i\chi_z} \; e^{i\nu - \frac{1}{2} \phi} ds, \]

\[ A_{1,1}(s) = -\frac{a_1}{\rho} + g(s)(\frac{\alpha_x - \alpha_z}{\beta_x + \beta_z}) + jg(s)(\frac{1}{\beta_x - \beta_z}), \]

\[ H = \nu x + \nu J_z + g \sqrt{J_x J_z} \cos(\psi - \chi) \cos \psi \quad g = |G_{1,1,1,1}| \]

\[ F_2 = (\psi + \psi - \ell \theta + \chi) I_1 + \psi J_2 \quad I_1 = J_x, \quad I_2 = J_z - J_x \]

\[ H = \delta \theta + g \sqrt{I_1(I_1 + I_2)} \cos \phi \cos \nu J_2, \quad \delta = (\nu_x + \nu_z - \ell) \]

\[ \ddot{I}_1 = [g^2 - \delta^2] I_1 + \frac{1}{2} (g^2 I_2 - \delta H) \]

\[ I_1 = a e^{g^2 \theta^2} + b e^{-g^2 \theta^2} + c + d \phi + \frac{1}{2} (\frac{1}{2} g^2 I_2 + \delta H) \phi^2 \]

\[ \sim a e^{2\pi \theta^2} \]

When the tunes ramp through resonances, the number of turns that the tune stays on resonance is \(\Delta n = g/(d\nu/dn)\). Thus we expect that the emittance growth is given by

\[ \text{EGF} = \exp \left[ \frac{\lambda \nu^2}{2 \nu d\nu/dn} \right] \]

In this calculation, the
tunes are ramped from
(6.85,7.80) to (5.85, 6.80)
Conclusion:

1. Systematic Nonlinear space charge resonances can be important in high intensity accelerators
2. For future neutron source design, one should try to avoid the systematic nonlinear space charge resonance, if it is possible!
3. For the non-scaling FFAG, the nonlinear resonances induced by the space charge potential can be the limiting factor. These resonances limit the phase advance of each basic cell to within $\pi/2$ to $\pi/3$, and thus the momentum acceptance is highly constrained.
4. I also find that the emittance growth factor for quadrupole and skew-quadrupole errors obeys a simple scaling law:

$$\text{EGF} = \exp \left[ \lambda \frac{2\pi g^2}{d\nu/dn} \right]$$